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Some Remarks on Cohen–Macaulay Rings with Many Zero Divisors and an Application

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INTRODUCTION

Let $R = k[X_1, ..., X_N]$ be a polynomial ring over a field k, and let $G \subset GL(N, k)$ be a finite group acting on R in the natural way. Assume that $(|G|, \operatorname{ch} k) = 1$, if $\operatorname{ch} k \neq 0$. Then, we have

THEOREM. *The ring of invariant R^G is regular (or what amounts to the same thing, the number of basic invariant forms is equal to N), if and only if G is generated by generalized reflexions.*

This was essentially proved by Chevalley [3] (and Shephard and Todd [12]) (cf. Serre [13]). In this paper, we prove this theorem via the following lemma: G is generated by generalized reflexions if and only if $R \otimes_{R^G} R$ is a Cohen–Macaulay ring. Intuitively, $\operatorname{Spec} R \otimes_{R^G} R = X$ can be thought of as the point set $\{(x, x^g) \mid x \in \operatorname{Spec} R, g \in G\}$, so that X is a union of vector spaces each of which is isomorphic to $\operatorname{Spec} R$. Thus, the lemma asserts that, in particular, in order for $R \otimes_{R^G} R$ to be Cohen–Macaulay, the "singularity" of X must be the largest possible.

It is the purpose of the present paper to show that this is the case for any Cohen–Macaulay ring A such that each component of A is a regular ring.

In the first section, we fix a regular local ring A and consider, mostly, ideals of the form $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_i$, where I is a finite set of indices and $\mathfrak{p}_i$ is a regular prime of A. (We say that $\mathfrak{p}$ is a regular prime if $A/\mathfrak{p}$ is regular.) Various properties of $A/\mathfrak{a}$ will be shown, especially when it is assumed to be a Cohen–Macaulay ring. Of course, our study of Cohen–Macaulay rings of this kind was motivated by Hochster's "polytopes of ideals" that played an important role in his papers [7, 8].

The second section is devoted to the proof of the above theorem and the lemma. The result is not new, but it is hoped that the proof here illustrates the properties of Cohen–Macaulay rings discussed in Section 1.

In Preliminaries we have collected some fundamental lemmas of homological algebra and have given some corollaries to them that will be used in the sequel.

In the following discussion, all rings are commutative, Noetherian, and with unit element, and modules are unitary. For a ring A, $\dim A$ denotes its Krull dimension.

Preliminaries

Proposition (1). *Let A be a regular local ring and M a finite module over A.*

(a) *Homological dimension of M over A, $\mathrm{hd}_A M$, is the greatest integer j such that $\mathrm{Ext}_A^j(M, A) \neq 0$.*

(b) *Grade of M is the least integer j such that $\mathrm{Ext}_A^j(M, A) \neq 0$. (Grade M is the maximal length of A-regular sequences contained in the annihilator of M.)*

Proof. See [1, Proposition 4.10] for (a) and [5, Proposition 3.3] for (b).

Remark. For an ideal $\mathfrak{a}$ of a Cohen–Macaulay ring A, the maximal length of A-regular sequences contained in $\mathfrak{a}$ equals the height of $\mathfrak{a}$. Hence, grade $A/\mathfrak{a} = \mathrm{ht}\,\mathfrak{a}$.

Proposition (2). *Let A and M be as above. M is a Cohen–Macaulay module if and only if grade $M = \mathrm{hd}_A M$, i.e., if and only if $\mathrm{Ext}_A^j(M, A) = 0$ for all but one j; and in this case, the j for which $\mathrm{Ext}_A^j(M, A) \neq 0$ is equal to grade M.*

Proof. It is well known that $\dim A = \mathrm{hd}_A M + \mathrm{depth}_A M$. On the other hand, $\dim M = \dim A/\mathrm{ann}\, M$, and grade $M = \mathrm{ht}\,\mathrm{ann}\, M$, so that we have $\dim A = \dim M +$ grade M. Therefore, depth $M = \dim M$ if and only if grade $M = \mathrm{hd}_A M$.

Corollary (3). *Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of finite modules over a regular local ring A. If M' and M'' are Cohen–Macaulay modules of equal grade g, M is also a Cohen–Macaulay module of grade g.*

Proof. Since $\mathrm{ann}\, M \supset \mathrm{ann}\, M'$ $\mathrm{Ext}_A^j(M, A) = 0$ for $j < g$ by Proposition (1)(b). On the other hand, we have the derived sequence $\mathrm{Ext}_A^j(M', A) \to \mathrm{Ext}_A^j(M, A) \to \mathrm{Ext}_A^{j+1}(M'', A)$. From Proposition (1)(a), it follows that $\mathrm{Ext}_A^j(M, A) = 0$ for $j > g$.

Corollary (4). *If M is a Cohen–Macaulay module of grade g, $\mathrm{Ext}_A^g(M, A)$ is also a Cohen–Macaulay module of grade g. Moreover, $\mathrm{Ass}\, M = \mathrm{Ass}\, \mathrm{Ext}_A^g(M, A)$.*

Proof. The first assertion is an easy consequence of Proposition (1). For the second assertion, observe that $\mathrm{Ext}_A{}^g(\mathrm{Ext}_A{}^g(M, A), A) \simeq M$. Then, clearly, $\mathrm{ann}\, M = \mathrm{ann}\, \mathrm{Ext}_A{}^g(M, A)$. For any Cohen–Macaulay module M, the set of associated primes coincides with that of minimal primes of $\mathrm{ann}\, M$. Hence, the assertion.

Let $\mathfrak{a}$ be an ideal of a regular local ring A such that $A/\mathfrak{a}$ is Cohen–Macaulay. We denote the canonical module of $A/\mathfrak{a}$ by $\Omega(A/\mathfrak{a})$. If grade $A/\mathfrak{a} = g$, $\Omega(A/\mathfrak{a})$ is isomorphic to $\mathrm{Ext}_A{}^g(A/\mathfrak{a}, A)$. Various properties of $\Omega(A/\mathfrak{a})$ are found in Herzog–Kunz [6] and Sharp [11]. In the latter, the term "a Gorenstein module of rank 1" is used for a canonical module. Recall that a Cohen–Macaulay ring of the form $A/\mathfrak{a}$ as above is Gorenstein if and only if $\Omega(A/\mathfrak{a}) = A/\mathfrak{a}$ (cf. Bass [2, Proposition 5.1]).

1. Cohen–Macaulay Rings Whose Components are Regular

Lemma (1.1). *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two arbitrary ideals of a ring* A. *Define morphisms* φ *and* ψ *as follows*: $\varphi\colon x \bmod \mathfrak{a} \cap \mathfrak{b} \mapsto (x \bmod \mathfrak{a}, x \bmod \mathfrak{b})$, $\psi\colon (x \bmod \mathfrak{a}, y \bmod \mathfrak{b}) \mapsto x - y \bmod \mathfrak{a} + \mathfrak{b}$. *Then*, $0 \to A/\mathfrak{a} \cap \mathfrak{b} \xrightarrow{\varphi} A/\mathfrak{a} \oplus A/\mathfrak{b} \xrightarrow{\psi} A/\mathfrak{a} + \mathfrak{b} \to 0$ *is an exact sequence.*

Proof. Straightforward.

The next proposition is due to Hochster and Eagon. They proved it using different exact sequences from the one given here (cf. [8, Proposition 17 and 18]).

Proposition (1.2). *Let* A *be a regular local ring and* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals such that both* $A/\mathfrak{a}$ *and* $A/\mathfrak{b}$ *are Cohen–Macaulay. Assume that* $A/\mathfrak{a}$ *and* $A/\mathfrak{b}$ *have the same dimension* d *and that* $\mathfrak{a}$ *and* $\mathfrak{b}$ *have no associated primes in common. Then, the following conditions are equivalent*:

(i) $A/\mathfrak{a} + \mathfrak{b}$ *is a Cohen–Macaulay ring of dimension* $d - 1$;

(ii) $A/\mathfrak{a} \cap \mathfrak{b}$ *is a Cohen–Macaulay ring.*

In this case, we have an exact sequence of canonical modules:

$$0 \to \Omega(A/\mathfrak{a}) \oplus \Omega(A/\mathfrak{b}) \to \Omega(A/\mathfrak{a} \cap \mathfrak{b}) \to \Omega(A/\mathfrak{a} + \mathfrak{b}) \to 0.$$

Proof. Because $\mathfrak{a}$ and $\mathfrak{b}$ have no associated primes in common, we know that $\mathrm{ht}(\mathfrak{a} + \mathfrak{b}) > \mathrm{ht}(\mathfrak{a})$. The assertion is then clear by Proposition (2) applied to the sequence of Lemma (1.1).

The sequence of (1.1) further enables us to compute Ext's in certain cases.

For example, let $\mathfrak{a}$ and $\mathfrak{b}$ be as in Proposition (1.2). Assume that $A/\mathfrak{a}+\mathfrak{b}$ is Cohen–Macaulay of dimension $d-c$ with $c>1$. Then,

$$\begin{aligned} \operatorname{Ext}_A{}^j(A/\mathfrak{a}\cap\mathfrak{b}, A) &\cong \Omega(A/\mathfrak{a})\oplus\Omega(A/\mathfrak{b}) && j=N-d\\ &\cong \Omega(A/\mathfrak{a}+\mathfrak{b}) && j=N-d+c-1\\ &\cong 0 && \text{otherwise,}\end{aligned}$$

where $N=\dim A$ so that $N-d=\operatorname{ht}\mathfrak{a}$. Prime ideals defined by linear equations in a polynomial ring are simple examples for $\mathfrak{a}$ and $\mathfrak{b}$.

COROLLARY (1.3). *In the situation of* (1.2), *if* $A/\mathfrak{a}\cap\mathfrak{b}$ *is Gorenstein, then* $A/\mathfrak{a}+\mathfrak{b}$ *is also Gorenstein.*

Proof. Immediate by Bass [2, Proposition 5.1].

Let $\mathfrak{a}=\bigcap_{i\in I}\mathfrak{q}_i$ be a shortest primary decomposition of an ideal $\mathfrak{a}$ in a ring A. Define $S(A/\mathfrak{a})$ as the cokernel of φ, which sends x mod $\mathfrak{a}$ to $(x \bmod \mathfrak{q}_i)_{i\in I}$: $0\to A/\mathfrak{a}\xrightarrow{\varphi}\bigoplus_{i\in I}A/\mathfrak{q}_i\to S(A/\mathfrak{a})\to 0$. With this notation, we have the following

LEMMA (1.4). *Assume that A is a regular local ring and that each $A/\mathfrak{q}_i$ is a Cohen–Macaulay ring. If $A/\mathfrak{a}$ is also Cohen–Macaulay and* $\operatorname{ht}\mathfrak{a}=h$, *then $S=S(A/\mathfrak{a})$ is a Cohen–Macaulay module of grade $h+1$. Hence, if P is an associated prime of S,* $\operatorname{ht}P=h+1$.

Proof. Since $\mathfrak{a}$ is unmixed, $\operatorname{ht}\mathfrak{q}_i=h$ for every $i\in I$. Hence, $A/\mathfrak{a}$ and $\bigoplus_{i\in I}A/\mathfrak{q}_i$ are Cohen–Macaulay modules of equal grade h. Hence, we have the exact sequence: $0\to\operatorname{Ext}_A{}^h(S,A)\to\operatorname{Ext}_A{}^h(\bigoplus_{i\in I}A/\mathfrak{q}_i, A)\to\operatorname{Ext}_A{}^h(A/\mathfrak{a},A)\to\operatorname{Ext}_A^{h+1}(S,A)\to 0$; the terms other than these are all 0. We have to show $\operatorname{Ext}_A{}^h(S,A)=0$. Suppose not. Then, it has an associated prime $\mathfrak{p}$, which is necessarily an associated prime of $\bigoplus_{i\in I}\operatorname{Ext}_A{}^h(A/\mathfrak{q}_i, A)$. Therefore, $\mathfrak{p}$ is the radical of one of $\mathfrak{q}_i$ (see Corollary (4)). But at each component $\mathfrak{p}$ of $A/\mathfrak{a}$, φ induces an isomorphism and $S_\mathfrak{p}=0$. Thus, we get a contradiction. The last assertion is clear by Proposition (1)(b).

Let A be a regular local ring. Let $\{\mathfrak{p}_i\}_{i\in I}$ be a finite set of prime ideals of A indexed by I. Assume $\operatorname{ht}\mathfrak{p}_i=h$ for all $i\in I$. Then, in general, $2h\geqslant\operatorname{ht}(\mathfrak{p}_i+\mathfrak{p}_j)>h$ if $i\neq j$. We define a symbol $\circ$ as follows: $\mathfrak{p}_i\circ\mathfrak{p}_j$ if and only if $\operatorname{ht}(\mathfrak{p}_i+\mathfrak{p}_j)=h+1$.

PROPOSITION (1.5). *Let A and $\mathfrak{p}_i$ be as above. Assume $A/\mathfrak{p}_i$ is Cohen–Macaulay for each $i\in I$. Let $\sim$ denote the equivalence relation generated by $\circ$. Put $\mathfrak{a}=\bigcap_{i\in I}\mathfrak{p}_i$. If $A/\mathfrak{a}$ is Cohen–Macaulay, then, for any $i,j\in I$, $\mathfrak{p}_i\sim\mathfrak{p}_j$.*

Proof. Let I_1 be the subset of I such that $i\in I_1$ if and only if $\mathfrak{p}_i\sim\mathfrak{p}_1$.

Assuming that $I - I_1 \neq \varnothing$, let $\mathfrak{a}_1 = \bigcap_{i \in I_1} \mathfrak{p}_i$ and $\mathfrak{a}_2 = \bigcap_{i \notin I_1} \mathfrak{p}_i$. Then, we have a homomorphism of exact sequences:

$$\begin{array}{ccccccccc} 0 \to & A/\mathfrak{a} & \to & A/\mathfrak{a}_1 \oplus A/\mathfrak{a}_2 & \to & A/\mathfrak{a}_1 + \mathfrak{a}_2 & \to 0 \\ & \downarrow \| \mathrm{id} & & \downarrow & & \downarrow & \\ 0 \to & A/\mathfrak{a} & \longrightarrow & \bigoplus_{i \in I} A/\mathfrak{p}_i & \longrightarrow & S & \longrightarrow 0. \end{array}$$

Since the vertical map at the middle is injective, so is the map on the right. This implies that if P is an associated prime of $A/\mathfrak{a}_1 + \mathfrak{a}_2$, P is an associated prime of S, and by Lemma (1.4), ht $P = h + 1$. Since P contains both $\mathfrak{a}_1$ and $\mathfrak{a}_2$, we get a contradiction, which completes the proof.

COROLLARY (1.6). *Let A and $\{\mathfrak{p}_i\}_{i \in I}$ be as above. Assume that each $\mathfrak{p}_i$ is a regular prime and that $A/\bigcap_{i \in I} \mathfrak{p}_i$ is Cohen–Macaulay. Then, there are elements f_i of A such that $\mathfrak{p}_i + \bigcap_{i \neq j} \mathfrak{p}_j = \mathfrak{p}_i + (f_i)$.*

Proof. $\mathfrak{p}_i + \bigcap_{i \neq j} \mathfrak{p}_j/\mathfrak{p}_i$ is an ideal of $A/\mathfrak{p}_i$, which is a UFD. As in the proof of (1.5), we see that $\mathfrak{p}_i + \bigcap_{i \neq j} \mathfrak{p}_j$ has no embedded primes and that the ideal $\mathfrak{p}_i + \bigcap_{i \neq j} \mathfrak{p}_j/\mathfrak{p}_i$ has height 1. Hence, the assertion follows.

This does not mean that $A/\bigcap_{i \neq j} \mathfrak{p}_i$ is Cohen–Macaulay even if $A/\bigcap_{i \in I} \mathfrak{p}_i$ is Cohen–Macaulay, but this is the case if it is Gorenstein. In fact:

PROPOSITION (1.7). *Let A be a regular local ring. Let $I = \mathfrak{a} \cap \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A of pure height h, having no associated primes in common. If A/I is Gorenstein, "$A/\mathfrak{a}$ is Cohen–Macaulay" implies that $A/\mathfrak{b}$ is Cohen–Macaulay.*

Proof. Consider the exact sequence $0 \to \mathfrak{a}/I \to A/I \to A/\mathfrak{a} \to 0$. According to Proposition (2) and Corollary (3), $\mathfrak{a}/I$ is a Cohen–Macaulay module and, therefore, $\mathrm{Ext}_A{}^h(\mathfrak{a}/I, A)$ is also Cohen–Macaulay and the sequence $0 \to \mathrm{Ext}_A{}^h(A/\mathfrak{a}, A) \to \mathrm{Ext}_A{}^h(A/I, A) \to \mathrm{Ext}_A{}^h(\mathfrak{a}/I, A) \to 0$ is exact. Assume A/I is Gorenstein. Then, $\mathrm{Ext}_A{}^h(A/I, A) = A/I$, which implies that $\mathrm{Ext}_A{}^h(\mathfrak{a}/I, A)$ is an image of A. Let $A/\mathfrak{b}' = \mathrm{Ext}_A{}^h(\mathfrak{a}/I, A)$. We are going to show that $A/\mathfrak{b}' = A/\mathfrak{b}$. First note that $\mathrm{Ass}(A/I) = \mathrm{Ass}(A/\mathfrak{a}) \coprod \mathrm{Ass}(A/\mathfrak{b})$ and that $\mathrm{Ass}(A/\mathfrak{b}) = \mathrm{Ass}(\mathfrak{a}/I)$. Then, by Corollary (4), $\mathrm{Ass}(A/\mathfrak{b}) = \mathrm{Ass}(A/\mathfrak{b}')$. Let $\mathfrak{p}$ be a minimal prime of I. If $\mathfrak{p} \in \mathrm{Ass}(A/\mathfrak{b})$, $A_\mathfrak{p}/\mathfrak{b}'A_\mathfrak{p} = \mathrm{Ext}_A{}^h(\mathfrak{a}/I, A)_\mathfrak{p} \cong \mathrm{Ext}^h_{A_\mathfrak{p}}(A_\mathfrak{p}/IA_\mathfrak{p}, A_\mathfrak{p}) \cong A_\mathfrak{p}/IA_\mathfrak{p}$; hence, $\mathfrak{b}'A_\mathfrak{p} = IA_\mathfrak{p} = \mathfrak{b}A_\mathfrak{p}$. If $\mathfrak{p} \in \mathrm{Ass}(A/\mathfrak{a})$, $\mathfrak{b}'A_\mathfrak{p} = A_\mathfrak{p}$. This proves that $\mathfrak{b}' = \mathfrak{b}$, because an unmixed ideal is an intersection of its primary components.

EXAMPLE. Let X_{ij} be planes defined by $x_i = x_j = 0$ in four-dimensional affine space with coordinates x_i, $i = 1, 2, 3, 4$. Consider $X = X_{12} \cup X_{23} \cup X_{34}$.

X is Cohen–Macaulay, but it cannot be Gorenstein since $X_{12} \cup X_{34}$ is not Cohen–Macaulay by Proposition (1.2).

LEMMA (1.8). *Let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_i$ be an irredundant intersection of regular primes. Assume that depth $A/\mathfrak{p}_i \geqslant 2$ for every $i \in I$. Then, the following conditions are equivalent:*

(i) *depth $A/\mathfrak{a} \geqslant 2$;*

(ii) *there is an $A/\mathfrak{a}$-regular element x such that $\bigcap_{i \in I} \mathfrak{p}_i + (x) = \bigcap_{i \in I} [\mathfrak{p}_i + (x)]$.*

Proof. (ii) $\Rightarrow$ (i) is clear because depth $A/\mathfrak{a} + (x) \geqslant 1$. (i) $\Rightarrow$ (ii). Let $S = S(A/\mathfrak{a})$ be as before: $0 \to A/\mathfrak{a} \to \bigoplus_{i \in I} A/\mathfrak{p}_i \to S \to 0$. Since depth $A/\mathfrak{a} \geqslant 2$ and depth $\bigoplus_{i \in I} A/\mathfrak{p}_i \geqslant 2$, it follows that depth $S \geqslant 1$. Take x to be a regular element for both $A/\mathfrak{a}$ and S. Then, we have the diagram with rows exact:

$$\begin{array}{ccc} 0 \to A/\mathfrak{a} + (x) & \longrightarrow & \bigoplus_{i \in I} A/\mathfrak{p}_i + (x) \\ \downarrow & & \downarrow \| \\ 0 \to A/\bigcap_{i \in I} [\mathfrak{p}_i + (x)] & \to & \bigoplus_{i \in I} A/\mathfrak{p}_i + (x). \end{array}$$

The diagram shows that $\mathfrak{a} + (x) = \bigcap [\mathfrak{p}_i + (x)]$.

PROPOSITION (1.9). *Let A be a regular local ring. Let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_i$ be an irredundant intersection of regular primes. If the following conditions are satisfied, then $A/\mathfrak{a}$ is a Cohen–Macaulay ring:*

(i) *$A/\mathfrak{a}$ is of pure dimension (say d), and*

(ii) *there is a sequence of elements in A, $x_1, \ldots, x_{d-1}$, such that $A/\mathfrak{a} \otimes_A A/(x_1, \ldots, x_r)$ is of pure dimension $d - r$ and such that $\mathfrak{a} + (x_1, \ldots, x_r)$ is an intersection of regular primes for $0 < r < d$.*

Proof. Note that if one-dimensional local ring is reduced it is Cohen–Macaulay. Then, by induction on d, we only have to show that if $d > 1$, x_1 is a regular element for $A/\mathfrak{a}$. But this is trivial, for unless it is so, $A/\mathfrak{a} \otimes A/(x_1)$ cannot be of pure dimension $d - 1$.

The converse of this proposition holds if the residue field of A contains infinitely many elements.

PROPOSITION (1.10). *Let $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_i$ be as in (1.9). Let $\mathfrak{m}$ be the maximal ideal of A. Assume $A/\mathfrak{m}$ is an infinite field. If $A/\mathfrak{a}$ is Cohen–Macaulay of dimension d, there is a maximal $A/\mathfrak{a}$-regular sequence $x_1, \ldots, x_d$ satisfying the following conditions:*

(i) $\mathfrak{a} + (x_1, ..., x_r)$ *is an intersection of regular primes* (*hence*, $A/\mathfrak{a} \otimes_A A/(x_1, ..., x_r)$ *is reduced*) *for* $r < d$.

(ii) *The length* $l(A/\mathfrak{a} \otimes A/(x_1, ..., x_d))$ *is equal to the number of components of* $A/\mathfrak{a}$, *i.e., the cardinality of* I.

Proof. Recall that $\mathfrak{p}$ is a regular prime if and only if $\mathfrak{p}$ is generated by a subset of a regular system of parameters of A, i.e., generated by a subset of representatives of a basis for the vector space $\mathfrak{m}/\mathfrak{m}^2$. Assume first that $\dim A/\mathfrak{a} = 1$. Then, since $A/\mathfrak{m}$ is infinite, it is possible to find an element x such that $\mathfrak{p}_i + (x) = m$ for every $i \in I$. From the exact sequence as in Lemma (1.4) $0 \to A/\mathfrak{a} \to \bigoplus_{i \in I} A/\mathfrak{p}_i \to S \to 0$, we obtain the exact sequence

$$0 \to \mathrm{Tor}_1{}^A(S, A/(x)) \to A/\mathfrak{a} \otimes_A A/(x) \to \bigoplus_{i \in I} (A/\mathfrak{p}_i \otimes_A A/(x))$$
$$\to S \otimes_A A/(x) \to 0.$$

This shows that $l(A/\mathfrak{a} + (x)) - \#I = l(S \otimes_A A/(x)) - l(\mathrm{Tor}_1{}^A(S, A/(x)))$. On the other hand, we have an exact sequence

$$0 \to \mathrm{Tor}_1{}^A(S, A/(x)) \to S \xrightarrow{x} S \to S \otimes_A A/(x) \to 0.$$

Thus, $l(S \otimes_A A/(x)) = l(\mathrm{Tor}_1{}^A(S, A/(x)))$, and $l(A/\mathfrak{a} + (x)) = \#I$. (Note that $l(S) < \infty$ since $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_i$ is irredundant.) Assume that $\dim A/\mathfrak{a} \geqslant 2$. Then, by the same reason as above, there is an $A/\mathfrak{a}$-regular element x such that $\mathfrak{p}_i + (x)$ is a regular prime for every $i \in I$ and such that it is an S-regular element. Then, by Lemma (1.8) $\mathfrak{a} + (x) = \bigcap_{i \in I} [\mathfrak{p}_i + (x)]$ and the proof is complete by induction on $\dim A/\mathfrak{a}$. We should note that the intersection $\bigcap_{i \in I} [\mathfrak{p}_i + (x)]$ is irredundant because x is an S-regular element.

2. An Application

Lemma (2.1). *Let R be a regular local ring and A a local ring that contains R. Assume that A is a finite module over R. Then, A is Cohen–Macaulay if and only if A is flat over R.*

Proof. See [10, Theorem (46)], or use the formulas: $\mathrm{hd}_R A + \mathrm{depth}_R A = \dim R$, and $\mathrm{depth}_R A = \mathrm{depth}_A A$. The first one is well known; for the second, see [9, Lemma 2], for example.

Lemma (2.2). *Let R' be a subring of a regular local ring R such that R is a finite R'-module. Then, R' is regular if and only if R is flat over R'.*

Proof. R' is a local ring, for, R is integral over R'. Let $\mathfrak{m}'$ be the maximal ideal of R'. Since R is regular, $\mathrm{hd}_R(R \otimes_{R'} R'/\mathfrak{m}') < \infty$. Assume R is flat over R'. Then, it is faithfully flat over R' and it follows that $\mathrm{hd}_{R'} R'/\mathfrak{m}' < \infty$, which is equivalent to that R' is regular. The converse is a special case of Lemma (2.1).

LEMMA (2.3). *Let R be an integral domain. Let R' be a subring of R such that the field of quotients of R is a separable extension over that of R'. Assume that, for any element $0 \neq x \in R$, $x \otimes 1$ and $1 \otimes x$ are not zero divisors in $R \otimes_{R'} R$. Then, $R \otimes_{R'} R$ is reduced.*

Proof. Let R_0 and R_0' denote the fields of quotients of R and R', respectively. Put $A = R \otimes_{R'} R$. Note that R is a subring of A. Since no element of R is a zero divisor of A, A is contained in $R_0 \otimes_{R'} R_0 = R_0 \otimes_{R_0'} R_0$, which is reduced, for, $R_0' \to R_0$ is a separable extension of fields.

In the following, R always denotes an integral domain and G a finite group acting on R as automorphisms of R. Let R^G denote the subring of R formed by G-invariant elements. We will always assume that the order of G is invertible in R, so that R^G is Noetherian over which R is a finite module. Moreover, the existence of the Reynolds operator $\rho: R \to R^G$, $\rho(x) = (1/|G|)\sum_{g\in G} x^g$, guarantees the following

LEMMA (2.4). *Let S be any R^G-algebra. Regard S as a subring of $R \otimes_{R^G} S$, and let G naturally act on $R \otimes_{R^G} S$ by S-automorphisms. Then, $(R \otimes_{R^G} S)^G = S$.*

Proof. See [4, Lemma (5.6)].

Let R, G, R^G, be as above. Put $A = R \otimes_{R^G} R$. We make A an R-algebra via $\lambda: R \to A$ defined by $\lambda(x) = 1 \otimes x$. Let G operate on A on the first factor of $R \otimes_{R^G} R$ so that $g \in G$ induces an R-automorphism of A. For each $g \in G$, define a morphism $\epsilon_g: R \otimes_{R^G} R \to R$, by $\epsilon_g(x \otimes y) = xy^g$, and let $\mathfrak{p}_g$ be the kernel of ϵ_g. $\mathfrak{p}_g$ is generated by the set $\{x^g \otimes 1 - 1 \otimes x \mid x \in R\}$, and $A/\mathfrak{p}_g$ is isomorphic to R. Roughly, $\operatorname{Spec} A = \bigcup_{g\in G} \operatorname{Spec} A/\mathfrak{p}_g$, and identifying $\operatorname{Spec} R$ with $\operatorname{Spec} A/\mathfrak{p}_e$ (e is the identity of G), $\operatorname{Spec} A/\mathfrak{p}_e \cap \operatorname{Spec} A/\mathfrak{p}_g$ is the "points fixed by g." Throughout the rest of this section, these notations will be fixed.

LEMMA (2.5). $\mathfrak{a} = \bigcap_{g\in G} \mathfrak{p}_g$ *consists of nilpotent elements.*

Proof. Let $x \in \mathfrak{a}$ and consider the polynomial $f(T) = \prod_{g\in G}(T - x^g)$. The coefficients of f are all invariant under the action of G and they are elements of $\mathfrak{a}$ as well. Lemma (2.4) says that $A^G = R$. It is obvious that $\mathfrak{a} \cap R = 0$. Thus, if the order of G is n, then $f(T) = T^n$, and $f(x) = 0$.

PROPOSITION (2.6). *Assume that R is a regular local ring with maximal ideal $\mathfrak{m}$ and that G is acting trivially on $R/\mathfrak{m}$. Then, A is a local ring. Moreover, A is a Cohen–Macaulay local ring if and only if R^G is a regular local ring.*

Proof. Note that $R^G/\mathfrak{m}^G = (R/\mathfrak{m})^G$. This is true because we have the Reynolds operator. Since A is finite over R, there are only a finite number of ideals in A and they all lie over $\mathfrak{m}$. But $A \otimes_R R/\mathfrak{m} = R \otimes_{R^G} R \otimes_R R/\mathfrak{m} = R \otimes_{R^G} R/\mathfrak{m} = R \otimes_{R^G} R^G/\mathfrak{m}^G$, and the last ring is clearly a local ring.

For the second assertion, consider the diagram:

$$\begin{array}{ccc} R & \xrightarrow{\lambda_1} & R \otimes_{R^G} R = A \\ \uparrow & & \uparrow \lambda \\ R^G & \xrightarrow{j} & R, \end{array}$$

where λ_1 is defined by $\lambda_1(x) = x \otimes 1$, and j is the inclusion map. The λ_1 is a morphism of G-modules R and A, and if we take invariant, it retracts to $j: R^G \to R$. By Lemmas (2.1) and (2.2), it suffices to show that, if λ_1 is flat, then j is flat (the converse being clear). If λ_1 is flat, then A is a free module over $\lambda_1(R)$, and A is generated, as a $\lambda_1(R)$-module, by the set $\lambda(R)$. Hence, we can choose a free basis of A over $\lambda_1(R)$, $e_1, \ldots, e_u$, in $\lambda(R)$. Write $A = \lambda_1(R)e_1 \oplus \cdots \oplus \lambda_1(R)\, e_u$ as a direct sum. Each $\lambda_1(R)\, e_i$ is a G-module, so that taking invariant we get $R = R^G e_1 \oplus R^G e_2 \oplus \cdots \oplus R^G e_u$. This shows that R is a free module over R^G.

LEMMA (2.7). *Assume that A is a Cohen–Macaulay local ring and that R is a regular local ring. Then, A is reduced and hence, $\mathfrak{a} = \bigcap_{g \in G} \mathfrak{p}_g = 0$.*

Proof. Note that the quotient field of R is separable over that of R^G (in fact it is a Galois extension with Galois group G). Since A is flat over R, no element of R is a zero divisor in A and A is reduced by Lemma (2.3).

DEFINITION I. Let V be a finite-dimensional vector space over a field k and let $GL_k(V)$ denote the group of invertible linear transformations of V over k. An element in $GL_k(V)$ of finite order is called a generalized reflexion if it pointwise fixes a subspace of V of codimension 1.

Following Hochster [8, p. 1034], we define a generalized reflexion as an automorphism of a ring as follows:

DEFINITION II. Let $g: R \to R$ be an automorphism of a ring R. Assume that g is of finite order, i.e., $g^n =$ identity for some n. Then, g is called a generalized reflexion if there is a nonunit element l of R such that $x - x^g \in (l)$ for all $x \in R$.

Remark 1. Suppose R is a polynomial ring over a field. When an automorphism g of R is given by a linear transformation of the variables of R, the two definitions above are consistent with each other. Geometrically, g is a generalized reflexion if and only if the induced action of g on Spec R pointwise fixes a hyperplane.

Remark 2. Definition I is not actually used as far as Proposition (2.15), where we relate our result Corollary (2.13) to that of Serre [13]. In [13], a generalized reflexion in the sense of Definition I is called a pseudoreflexion.

THEOREM (2.8). *Assume that R is a regular local ring and that G is acting trivially on the residue field of R. If A is Cohen–Macaulay, G is generated by generalized reflexions.*

Proof. Let e denote the identity element of G. Then, it is easy to see that $g \in G$ is a generalized reflexion if and only if $\mathrm{ht}(\mathfrak{p}_e + \mathfrak{p}_g) = 1$, for, with the identification $A/\mathfrak{p}_e \xrightarrow{\sim} R$, $A/\mathfrak{p}_e + \mathfrak{p}_g$ is isomorphic to R/I, where I is the ideal generated by the set $\{x - x^g \mid g \in G, x \in R\}$. Since A is finite over R, A is a quotient of a regular local ring by an ideal that is an intersection of regular primes (Lemmas (2.6) and (2.7)). By Proposition (1.5), we get the conclusion: To be precise, let H be the subgroup of G generated by the generalized reflexions of G. Set $\mathfrak{a}_1 = \bigcap_{h \in H} \mathfrak{p}_h$ and $\mathfrak{a}_2 = \bigcap_{g \notin H} \mathfrak{p}_g$. If $\mathfrak{a}_2 \neq A$, as in the proof of (1.5), there is a prime P of height 1 containing both $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Then, P contains a $\mathfrak{p}_h$ with $h \in H$, and a $\mathfrak{p}_g$ with $g \in H$. Since $\mathfrak{p}_h + \mathfrak{p}_g$ is isomorphic to $\mathfrak{p}_{hg^{-1}} + \mathfrak{p}_e$, $hg^{-1} \in H$. This is a contradiction.

The homogeneous version of the theorem is as follows:

COROLLARY (2.9). *Let $R = k[X_1, \ldots, X_N]$ be a polynomial ring over a field k, and let G be a finite group acting on R by k-linear automorphisms. Assume $(|G|, \mathrm{ch}\, k) = 1$ if $\mathrm{ch}\, k \neq 0$. If $A = R \otimes_{R^G} R$ is Cohen–Macaulay, G is generated by generalized reflexions.*

Remark. As will be seen below, if $G \subset GL(N, k)$ is generated by generalized reflexions, there are N invariant forms, $f_1, \ldots, f_N$, such that $R^G = k[f_1, \ldots, f_N]$. We have proved that, in Lemma (1.6), there is an f such that $\mathfrak{p}_e + \bigcap_{e \neq g} \mathfrak{p}_g = \mathfrak{p}_e + (f)$. This f is, of course, the determinant of the Jacobian matrix of f_i's.

THEOREM (2.10). *Assume that R is a regular local ring and that G is generated by generalized reflexions. Assume that the induced action of G on the residue field of R is trivial. Then, $A = R \otimes_{R^G} R$ is a Cohen–Macaulay local ring.*

To avoid complexity, we state a part of the proof as a lemma:

LEMMA (2.11). *Let B be a local ring acted upon by a finite group G that is generated by generalized reflexions. Let G_0 be a subset of G consisting of generalized reflexions that generate G. For each $g \in G_0$, let l_g be a nonunit element of R satisfying $x - x^g \in (l_g)$ for all $x \in R$. Let $\mathfrak{p}$ be a prime ideal of B such that, (a) $\mathfrak{p} \cap B^G = 0$, (b) $l_g{}^h \notin \mathfrak{p}$ for all $g \in G_0$ and for all $h \in G$. Then, $\bigcap_{g\in G} \mathfrak{p}^g = 0$, where $\mathfrak{p}^g$ denotes the image of $\mathfrak{p}$ by the automorphism g.*

Proof. Let $\mathfrak{n}$ be the maximal ideal of B and let $\mathfrak{a} = \bigcap_{g\in G} \mathfrak{p}^g$. We are going to show that $\mathfrak{a} = \mathfrak{n}\mathfrak{a}$, which proves that $\mathfrak{a} = 0$ by Nakayama's lemma. Let $a \in \mathfrak{a}$. If $g \in G_0$, there is an element b such that $a - a^g = l_g b$. Since $\mathfrak{a}$ is G-stable, $a - a^g = l_g b \in \mathfrak{a}$. The condition (b) is equivalent to saying that l_g is a regular element of $B/\mathfrak{a}$, i.e., $\mathfrak{a}: l_g = \mathfrak{a}$ for all $g \in G_0$. Hence, it follows that $b \in \mathfrak{a}$, and so $a - a^g \in \mathfrak{n}\mathfrak{a}$. Since G is generated by G_0 and since $\mathfrak{n}\mathfrak{a}$ is G-stable, we see that $a - a^g \in \mathfrak{n}\mathfrak{a}$ for all $g \in G$. Therefore, if n is the order of G, $na - \sum_{g\in G} a^g \in \mathfrak{n}\mathfrak{a}$. By the condition (a), we have $\sum_{g\in G} a^g \in \mathfrak{a} \cap B^G \subset \mathfrak{p} \cap B^G = 0$, which proves $a \in \mathfrak{n}\mathfrak{a}$. (We are assuming n is invertible in B.)

Proof of Theorem. As above, let G_0 be a set of generalized reflexions that generates G. For each $g \in G_0$, fix l_g, which has the property stated in the lemma. Let $N = \dim R$. Then, we can find $N - 1$ elements of R, $x_1, \dots, x_{N-1}$, such that $(x_1, \dots, x_{N-1})$ is a regular prime of height $N - 1$ and such that $l_g{}^h \notin (x_1, \dots, x_{N-1})$ for all $g \in G_0$ and for all $h \in G$. Put $P_r = (x_1, \dots, x_r)$, $r = 0, 1, \dots, N - 1$. We are going to show that, for all r, $A \otimes_R R/P_r \cong R \otimes_{R^G} R/P_r$ is reduced of pure dimension $N - r$, which proves that $x_1, \dots, x_{N-1}$ is a regular sequence for A and that $A \otimes_R R/P_{N-1} \cong A/(x_1, \dots, x_{N-1})A$ is a Cohen–Macaulay ring (for it is one-dimensional and reduced). Then, we may conclude that A itself is Cohen–Macaulay. (cf. Proposition (1.9)).

Set $P = P_r$. Let $\mathfrak{p}$ be the kernel of the morphism $\varphi: R \otimes_{R^G} R/P \to R/P$ defined by $\varphi(x \otimes \bar{y}) = \bar{x}\bar{y}$, where $-$ denotes the residue class modulo P. Observe that:

(i) the action of $g \in G$ on $R \otimes_{R^G} R/P$ is such that $(x \otimes \bar{y})^g = x^g \otimes \bar{y}$, and with this action of G on $R \otimes_{R^G} R/P$, G_0 still consists of generalized reflexions. In fact, if $g \in G_0$, and if $\xi = \sum_\alpha x_\alpha \otimes y_\alpha$ is an arbitrary element of $R \otimes_{R^G} R/P$, then $\xi - \xi^g = (l_g \otimes \bar{1})(\sum_\alpha z_\alpha \otimes \bar{y}_\alpha)$, where z_α are elements of R satisfying $x_\alpha - x_\alpha{}^g = l_g z_\alpha$.

(ii) $(R \otimes_{R^G} R/P)^G = R/P$ by Lemma (2.4), and $\mathfrak{p} \cap R/P = 0$.

(iii) For all $g \in G_0$ *and for all* $h \in G$, $l_g{}^h \otimes 1 \notin \mathfrak{p}$.

Apply the above lemma with $B = R \otimes_{R^G} R/P$ to conclude $\bigcap_{g\in G} \mathfrak{p}^g = 0$.

Q.E.D.

COROLLARY (2.12). *Let $R = k[X_1, \dots, X_N]$ and let G be as in (2.9). If G is generated by generalized reflexions, $R \otimes_{R^G} R$ is a Cohen–Macaulay ring.*

COROLLARY (2.13). *Let G be a finite group acting on a regular local ring R. Assume G is acting trivially on the residue field of R. Then, R^G is regular if and only if G is generated by generalized reflexions.*

Proof. Immediate by Theorems (2.8) and (2.10) and Proposition (2.6).

COROLLARY (2.14). *Assume $G \subset GL(N, k)$ is a finite group naturally acting on $k[X_1, \ldots, X_N] = R$. Assume $(|G|, \operatorname{ch} k) = 1$ if $\operatorname{ch} k \neq 0$. Then, the number of basic invariant forms is equal to N if and only if G is generated by generalized reflexions.*

Proof. Note that R^G is a graded ring finitely generated over k, so that it is a quotient of a polynomial ring by a homogeneous ideal. Then it is easy to see that R^G is regular if and only if R^G is a polynomial ring. We have already proved that R^G is regular if and only if $R \otimes_{R^G} R$ is a Cohen–Macaulay ring. (cf. Serre [13]) Q.E.D.

THEOREM (proved by Serre). *Let R and G be as in Corollary* (2.13). *Let $\epsilon: G \to GL(m/m^2)$ be the representation of G to the Zariski tangent space of R (which is gotten in the obvious fashion). Then, R^G is regular if and only if $\epsilon(G)$ is generated by generalized reflexions.*

For completeness, we show that this theorem and Corollary (2.13) coincide as they should. Namely, we prove:

PROPOSITION (2.15). *Let $g: R \to R$ be an automorphism of a regular local ring with maximal ideal $\mathfrak{m}$. Assume that g is of finite order (i.e., $g^n =$ identity for some n) and that the order is invertible in R. Moreover, assume that g induces an identity map on the residue field of R. Then, the following two conditions are equivalent:*

(i) *g is a generalized reflexion;*

(ii) *$\epsilon(g) \in GL(\mathfrak{m}/\mathfrak{m}^2)$ is a generalized reflexion.*

(See Definitions I and II.)

We need an easy lemma.

LEMMA (2.16). *With the same assumptions as above, $\epsilon(g) =$ identity implies $g =$ identity.*

Proof. Let G be the finite group of automorphisms of R generated by the single element g. Let $\rho: R \to R^G$ be the Reynolds operator. If $\epsilon(g)$ is identity, then for any $x \in R$, $x = \rho(x) \bmod \mathfrak{m}^2$ so that $\mathfrak{m}$ is generated by invariant elements, i.e., $(\mathfrak{m} \cap R^G)R = \mathfrak{m}$. Moreover, R is finite over R^G. By Nakayama's lemma, $R = R^G$ and g is identity.

Proof of Proposition. (ii) $\Rightarrow$ (i). Let I be the ideal of R generated by $\{x - x^g \mid x \in R\}$. Then, it is easy to see that g is a generalized reflexion if and only if $\operatorname{ht} I = 1$, for, since R is a UFD, an ideal of height 1 is contained in a principal ideal. Hence, we may assume that R is complete. Let G be as in the proof of the lemma. From the complete reducibility of the representation $\epsilon: G \to GL(\mathfrak{m}/\mathfrak{m}^2)$, it immediately follows that, if $\epsilon(g)$ is a generalized reflexion of order n, there is a basis $\bar{x}_1, \ldots, \bar{x}_N$ for $\mathfrak{m}/\mathfrak{m}^2$ such that

$$\begin{aligned} \epsilon(g)\, \bar{x}_1 &= \bar{c}\bar{x}_1 , \\ \epsilon(g)\, \bar{x}_2 &= \bar{x}_2 , \\ &\cdots \\ &\cdots \\ \epsilon(g)\, \bar{x}_N &= \bar{x}_N , \end{aligned}$$

where $\bar{c}$ is a primitive nth root of unity in $k = R/\mathfrak{m}$. Since R is complete, a representative c of $\bar{c}$ can be chosen also to be a primitive nth root of unity. Since g is identity mod $\mathfrak{m}$, it must hold that $c^g = c$. (In fact $(c^g)^n = 1$, hence, $c^g = c^m$ with $m < n$. Hence, $c - c^g = c(1 - c^{m-1})$. $c = c^g \bmod \mathfrak{m}$ implies $1 = c^{m-1} \bmod \mathfrak{m}$, so, if $m \neq 1$, it is a contradiction.) Let x be a representative of $\bar{x}_1$ and consider the element

$$l = c^{n-1}x + c^{n-2}x^g + c^{n-3}x^{g^2} + \cdots + cx^{g^{n-2}} + x^{g^{n-1}}.$$

Note that l is a semiinvariant of g so that g induces an automorphism $\bar{g}$: $R/(l) \to R/(l)$; In fact,

$$l^g = c^{n-1}x^g + c^{n-2}x^{g^2} + \cdots cx^{g^{n-1}} + x = cl.$$

We claim that $\bar{g}$ is identity, which is obviously equivalent to saying that $x - x^g \in (l)$ for all $x \in R$. By the above lemma, it suffices to show that the action of $\bar{g}$ on the tangent space of $R/(l)$ is trivial. But, since $l = nc^{n-1}x \bmod \mathfrak{m}^2$, the tangent space of $R/(l)$ is canonically isomorphic to $\mathfrak{m}/\mathfrak{m}^2 + (x)$, on which the subspace $k\bar{x}_2 + k\bar{x}_3 + \cdots + k\bar{x}_N$ of $\mathfrak{m}/\mathfrak{m}^2$ projects, and the assertion is clear.

(i) $\Rightarrow$ (ii). Assume that there is l such that $x - x^g \in (l)$ for all $x \in R$. The (l) is a G-stable ideal, and on $R/(l)$, G acts trivially. We may assume $l \in \mathfrak{m} - \mathfrak{m}^2$, for otherwise, g is identity. Let V and V' be the tangent spaces of R and $R/(l)$, respectively. Then, $\dim V = \dim V' + 1$, and as above, we have the exact sequence of G-modules: $0 \to k \to V \to V' \to 0$. Hence, V contains a subspace of codimension 1 left invariant by G.

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